THE LAWS OF ELASTICO-VISCOUS FLOW. II[1]

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In a paper of Harold Jeffreys entitled "The Viscosity of the Earth"[2] the author makes use of a formula which combines the laws of Larmor and of Maxwell.

$$n(S+\tau_2\frac{ds}{dt}=F+\frac{1}{\tau_1}\int Fdt$$

The integral implies a permanent set which, as the author indicates, would be inconsistent with the "accepted theories of tidal friction and variation of latitude. Hence τ_1 must be practically infinite." The formula is thus reduced to the expression $F=n_1\ S+\tau_2\frac{ds}{dt}$.

Experiments made on a great variety of materials show, however, that this expression must be seriously modified to represent the facts.

Thus it has been shown[3] that the displacement produced by a stress P is given by the expression $S=C_1Pe^{h_1P}+C_2Pe^{h_2P}(1-e^{-a\sqrt{t}})+C_3Pe^{h_3t^\rho}$.[4] The last term produces permanent set, so that for the present it may be omitted. Putting $C_1=\frac{1}{n_1}$ and $C_2=\frac{1}{n_2}, a=\frac{1}{\tau}, \theta=\sqrt{t}$, this becomes

$$S=\frac{P}{n_1}\left[A_1+\frac{n_1A_2}{n_2}\left(1-e^{-\frac{\theta}{\tau}}\right)\right] \quad \text{whence}$$

$$n_1\left(S+\tau\frac{ds}{d\theta}\right)=P\left(A_1+\frac{n_1}{n_2}A_2\right) .$$

For small stresses $A_1 = A_2 = 1$, and if n_2 equal n_1 this expression takes a form resembling that given by Jeffreys.

It is important to note, however, that this formula is based on the assumption that the viscosity is "external," that is, it acts as though the viscous resistance were due to an absolute velocity $\dfrac{ds}{dt}$.

But this is by no means evident; and indeed the probability is that a considerable part if not the major part of the viscous resistance may be "internal," that is, due to the relative motion of parts. Thus if an element consists of two parts y and z, y being coupled to the next adjacent element by an elastic coupling n_1 and with z by an elastic coupling n_2, together with a viscous coupling e_2, while e_1' and e_2' represent the "external" viscosities, the equations of motion will be

$$\rho_1 \ddot{z} = e_2(\dot{z} - \dot{y}) + n_2(z - y) + e_1'\ddot{z}$$

$$\rho_2 \ddot{y} = e_2(\dot{z} -) \dot{y} + n_2(z - y) + e_2'\dot{y} + n_1\frac{d^2y}{dx^2}$$

If $\rho_2 e_1'$ and e_2' be considered negligible, the solution, for not too rapid extinction, is

$$z = ae^{-\beta x} \cos p(t - vx),$$

in which

$$v^2 = \frac{n_1}{\rho} \frac{(n_2 - \rho p^2)^2 + p^2 e^2}{n_2(n_2 - \rho p^2) + p^2 e^2}$$

$$\beta = \frac{\rho^{3/2} p^4 e}{2\sqrt{n_1[n_2(n_2 - \rho p^2) + p^2 e^2]}} \, .$$

If pe is large compared with n_2

$$v^2 = \frac{n}{\rho}\left(1 + \frac{\rho^2 p^2}{e^2}\right)$$

$$\beta = \frac{\rho p^2}{2ev},$$

so that in this case the higher the viscosity the less rapid the decay of the oscillations—quite the reverse of the conclusions on the former

assumption.   But the appearance of $\theta = \sqrt{t}$ is a more serious matter, making the use of the formula much more difficult.

The operator which should replace n is therefore

$$n_1 \frac{1 + 2\tau \sqrt{i} \dfrac{ds}{dt}}{1 + \dfrac{n_1}{n_2} A_2} .$$

But the application of this formula to such a problem as the earth's viscosity is still further complicated by the fact that all the constants are functions of the pressure and of the temperature in the earth's interior.   Even though more or less probable assumptions may be made regarding the value of temperature and pressure as functions of the distance from the center, we know but little regarding the effect of these factors on either rigidity or viscosity. It was found that the temperature effect may be represented with considerable accuracy by the expression

$$A = E e^{(K + bP)\, \theta} ,$$

in which P is the applied stress, θ the temperature, and E, K, and b constants.

For room temperature the values of $b\theta = h$ are given in Table IV.

If we take $h_2 = 0.2$ as fairly representative

$$S_2 = P e\, e^{.2P} \left(1 - e^{-\sqrt{i}} \right) .$$

The unit $P = 100$ gm., so that G the couple $= Pr$ gm. cm.   Thus we get for the displacement, after a sufficiently long time,

$$S = P e\, e^{.2P} \quad \text{and} \quad \frac{1}{n_2} = \frac{S}{P} .$$

Table I shows the very great increase in importance of the elastico-viscous term for large stresses. The same is also true for the purely viscous term.

TABLE I

P	$=$	S/P
0		1
1		1.2
10		7.
50		20,000.

Table II gives the ratio $\dfrac{n_2}{n_1}$ for twenty-two materials, showing that there are certainly two elasticities, one of which is not accompanied by viscosity and the second is thus affected. In every case excepting that of sealing wax, where the ratio is unity, the second elasticity is much greater than the first, and in some cases enormously greater.

TABLE II

	$\dfrac{n_2}{n_1}$		$\dfrac{n_2}{n_1}$
Tin	60	Slate	150
Zinc	70	Shale	50
Marble	45	Soapstone	35
Limestone	30	Lead	40
Ebonite	8	Cadmium	65
Iron	2,200	Gold	150
Steel	12,000	Magnesium	250
Copper	8	Bapelite	7
Aluminum	4,400	Ivory	50
Talc	200	Silver	80
Glass	100	Sealing wax	1

The introduction of $\sqrt{t}$ instead of t itself is a step so radical that it may be well to give an illustration in its justification. For this purpose it is desirable to choose a material in which the elastico-viscous effect is well marked. This is notably the case for vulcanite, which has the added advantage of the relatively small importance of the third or purely viscous term. This illustration is perhaps the most striking in showing the appropriateness of $\sqrt{t}$ instead of t; but all the materials investigated give similar results.

Table III is a table of results for R_0, the return at the time t after releasing the stress.[1] $R_{\sqrt{t}}$ gives the result of calculation from

$$R = 890\left(1 - e^{-.57\sqrt{t}}\right).$$

R_t gives values calculated from

$$R = 840\left(1 - e^{-.4t}\right).$$

The differences between calculated and observed values under Δ_1 and Δ_2 show that the former expression is very near the truth, while the latter is entirely inadequate.

TABLE III

t	R_0	$R_{\sqrt{t}}$	Δ_1	R_t	Δ_2
1..........	380	387	7	277	− 103
2..........	490	492	2	462	− 28
4..........	600	605	5	672	+ 72
9..........	730	729	−1	820	+ 90
16..........	800	802	2	838	+ 38
25..........	840	841	1	840	∞
30..........	853	851	−2	840	− 13
∞..........	890	890	0	840	− 50

While the term involving a permanent set may not have any application to the problem of the earth tides, yet it may not be amiss to draw attention to the fact that in some cases and especially at temperatures approaching the melting-point, this term becomes the most important of all. The temperature coefficient in this case enters in the form $\theta/T - \theta$, giving as it should perfect fluidity at T, the melting-point.

In the former article the expression given for this viscous term is $S_3 = (Ft)^\rho$, in which $F = C_3 P e^{h_3 P}$ and ρ is stated to be approximately one-half.

From more recent data the average value of ρ is .41; and if from the nineteen substances examined four be excluded the average is .35, which makes it much nearer one-third than one-half.

[1] It was found by experiment that for stresses not too great the "direct" curve (on applying the stress) and the "return" curve (on releasing) were the same; or rather if the former is S and the latter R, then $S + R = Ct$.

TABLE IV

	c_1	c_2	c_3	c_4	h_1	h_2	h_3	h_4	a	p
Tin slowly cooled	660	20.	3.0	1.0	.10	1.0	3.5	4.5	0.8	0.6
Tin quickly cooled	640	8.	0.6	0.3	.00	1.8	3.8	4.6	0.8	0.6
Zinc slowly cooled	312	3.	3.4	2.0	.00	0.4	0.3	0.8	1.0	0.5
Zinc quickly cooled	300	8.4	3.0	.00		0.4	0.6	1.8	1.0	0.5
Marble	840	18.	40.	34.	.14	1.8	0.8	0.3	1.1	0.2
Limestone	600	20.	9.	11.	.10	0.3	0.3	0.8	1.2	0.3
Ebonite first determ.	13×10^3	1600.	600?	0?	.00	0.1	0.0		0.6	0.5
Ebonite second determ.	13×10^3	1800.	750.		.00	0.1	0.2		0.9	0.7
Soft iron annealed	145	.06		.00	.00	0.1		0.1	0.9	0.2
Soft iron unannealed	155	.08	.00	.15	.04	0.2		0.1	1.0	0.2
Tool steel annealed	144	.01	.00	.15	.00			0.1	1.0	0.3
Tool steel glass hard	143	.14	.26	.10	.06	.00	.03	.06	0.4	0.4
Copper annealed	250	1.3	.00	.07	.00	0.1	1.8	1.2	0.4	0.3
Aluminum annealed	440	0.1		.75	.60	1.0	2.5	1.4	0.5	0.3
Talc parallel cleavage	1340	4.0	20.	.9	.01	3.0	1.4	1.2	1.0	0.3
Slate parallel cleavage	382	2.0	0.4	110.00	.93	0.3	0.7	0.6	1.0	0.3
Slate perpendicular to cleavage	400	4.0	1.6	0.6		0.3	.00	0.8	1.2	0.3
Sealing wax	3×10^4	2×10^4	?	1.0	.00	.00	.00	.00	1.5	0.6
Glass (plate)	446	3.0	.00	?	.00	.00	.00	.00	0.7	
Glass (lead)	425	5.6	.00	4.0	.02	.16	.35	.15	0.5	
Shale parallel cleavage	400	8.4	1.0	0.1	.00			0.6	1.4	
Lead	1500	40.	7.	1.2	.00	0.2	1.9	3.6	1.2	0.4
Cadmium	460	70.	5.4	4.	.00	0.2	0.6	0.8	1.1	0.4
Gold	450	3.	0.2	2.4	.00	.04	0.2	0.3	1.0	0.5
Magnesium	2300	9.4	8.	1.0	.00	.25	0.4	0.6	1.0	0.3
Bakellite	7400	960.	14.	50.	.00	.12	0.2		1.2	0.3
Ivory	10^4	320.	3.	.00		.10	0.2		0.8	0.8
Silver	480	6.	2.	0.4	.00	.06	0.1	0.2	1.0	0.3

The expression for the viscous term should be $S_3 = (Ft)^{\frac{1}{3}}$ if the stress (P) is constant.   If P is a function of time

$$S_3 = (\textstyle\int Fdt)^{\frac{1}{3}}.$$

Thus if P be given a constant value for a time t_0 and then changed to P_1 the corresponding value of the viscous term would be

$$S_3 = (F_0\, t_0 - F_1 t)^{\frac{1}{3}}.$$

If the first stress be considerable and act for a long time the effect of the second stress is negligible.

Table IV is a provisional table of the constants which appear in the formula for the torsional strain at room temperature.

$$S = A_1 + A_2\left(1 - e^{-a\sqrt{t}}\right) + A_3 t^p + A_4,$$

in which $A = CPe^{hP}$. P is the weight acting on a pulley of radius 5 cm., the unit of weight being 100 gms. and the unit of time one minute.   The specimen is a cylindrical rod 7.5 cm. long and 4 mm. in diameter.

The term A_4, which may be termed the "lost motion," should probably be considered as a part of the viscous term, but with a very small exponent r, so that the whole viscous term may be represented by

$$S_3 = C_3 Pe^{h_3 P}\left(t^p + Bt^r\right).$$